

Completely co-bounded Schur multipliers

by

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Abstract

A linear map $u: E \rightarrow F$ between operator spaces is called completely co-bounded if it is completely bounded as a map from E to the opposite of F . We give several simple results about completely co-bounded Schur multipliers on $B(\ell_2)$ and the Schatten class S_p . We also consider Herz-Schur multipliers on groups.

In this short note, we wish to draw attention to the notion of “completely co-bounded” mapping between two operator spaces. Recall that an operator space can be defined as a Banach space E given together with an isometric embedding $E \subset B(H)$ into the space $B(H)$ of all bounded operators on a Hilbert space H . The theory of operator spaces started around 1987 with Ruan’s thesis and has been considerably developed after that (notably by Effros-Ruan and Blecher-Paulsen, see [2, 8]), with applications mainly to Operator Algebra Theory. In this theory, the morphisms between operator spaces are the completely bounded maps (c.b. in short), defined as follows. First note that if $E \subset B(H)$ is any subspace, then the space $M_n(E)$ of $n \times n$ matrices with entries in E inherits the norm induced by $M_n(B(H))$. The latter space is of course itself equipped with the norm of single operators acting naturally on $H \oplus \cdots \oplus H$ (n times). Then, a linear map $u: E \rightarrow F$ is called completely bounded (c.b. in short) if

$$(1) \quad \|u\|_{\text{cb}} \stackrel{\text{def}}{=} \sup_{n \geq 1} \|u_n: M_n(E) \rightarrow M_n(F)\| < \infty$$

where, for each $n \geq 1$, u_n is defined by $u_n([a_{ij}]) = [u(a_{ij})]$. One denotes by $CB(E, F)$ the space of all such maps.

Given an operator space E , the opposite E^{op} is the same Banach space as E , but equipped with the operator space structure (o.s.s. in short) associated to any embedding $E \subset B(H)$ such that for any $a = [a_{ij}] \in M_n(E)$, we have $\|a\|_{M_n(B(H))} = \|[a_{ji}]\|_{M_n(E)}$. Thus $\|a\|_{M_n(E^{op})} = \|[a_{ji}]\|_{M_n(E)}$. It is easy to check that the (isometric linear) mapping $T \mapsto {}^tT \in B(H^*)$ realizes such an embedding (warning: here ${}^tT: H^* \rightarrow H^*$ designates the adjoint of T in the Banach space sense).

We call a map $u: E \rightarrow F$ between operator spaces completely co-bounded if the same map is c.b. from E to F^{op} . This definition is inspired by existing work on completely co-positive maps

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(cf. e.g. [4, 5] and references there). I started to think about this notion after hearing Marciniak's lecture on co-positive multipliers at the 2004 Quantum probability conference in Bedlewo.

While this definition seems at first glance a pointless variation, easy to reduce to the usual case, we hope in what follows to convince the reader that it has a natural place in operator space theory and that it suggests many interesting questions. As a first motivation for this notion, we should mention that the non-commutative Grothendieck theorem, that came out of work by the author and Haagerup, can be rephrased as saying that, if A, B are C^* -algebras, any bounded linear mapping $u: A \rightarrow B^*$ is the sum of a c.b. mapping and a co-c.b. one, see [9, p. 189] for details and more references.

Definition 1. A linear map $u: E \rightarrow F$ between operator spaces will be called completely co-bounded if it is completely bounded as a mapping from E into F^{op} the opposite operator space. We then denote

$$\|u\|_{cob} = \|u: E \rightarrow F^{op}\|_{cb}.$$

Remark 2. Obviously, $\|u: E \rightarrow F^{op}\|_{cb} = \|u: E^{op} \rightarrow F\|_{cb}$ and $(F^{op})^* = F^{*op}$ completely isometrically. Therefore, $u: E \rightarrow F$ is completely co-bounded iff the same is true for u^* and $\|u\|_{cob} = \|u^*\|_{cob}$, since this is valid for c.b. maps (cf. e.g. [2, 8]).

Remark 3. Clearly if B is a C^* -algebra with $F \subset B$ and if $\alpha: B \rightarrow B$ is an anti-automorphism (for instance transposition on $B(\ell_2)$), then $u: E \rightarrow F$ is completely co-bounded iff αu is c.b. and $\|u\|_{cob} = \|\alpha u\|_{cb}$. It is well known that the transposition on M_n has c.b. norm equal to n (cf. e.g. [8, p. 418-419]). Therefore, the identity map on $B(H)$ is *not* completely co-bounded unless H is finite dimensional. More generally, the identity map on a von Neumann algebra B is completely co-bounded iff B is of type I_n for some finite n , i.e. iff B is a direct sum of finitely many algebras of the form $M_n \otimes A_n$, with A_n commutative.

At first glance, the reader may have serious doubts for the need of the preceding notion ! but hopefully the next result will provide some justification.

Theorem 4. A Schur multiplier $M_\varphi: [x_{ij}] \rightarrow [\varphi_{ij}x_{ij}]$ is completely co-bounded on $B(\ell_2)$ iff the matrix $[[\varphi_{ij}]]$ defines a bounded operator on ℓ_2 and we have

$$(2) \quad \|M_\varphi\|_{cob} = \|TM_\varphi\|_{cb} = \|[[\varphi_{ij}]]\|_{B(\ell_2)}$$

where $T: B(\ell_2) \rightarrow B(\ell_2)$ denotes the transposition. Moreover, if $\|M_\varphi\|_{cob} \leq 1$, then M_φ admits a factorization

$$B(\ell_2) \xrightarrow{J} \ell_\infty(\mathbb{N} \times \mathbb{N}) \xrightarrow{M_\varphi} B(\ell_2)$$

where J is the natural inclusion map and where $\|M_\varphi\|_{cb} \leq 1$.

Proof. Assume that $[[\varphi_{ij}]]$ is in $B(\ell_2)$. Then the mapping

$$\begin{aligned} \mathcal{M}_\varphi: \ell_\infty(\mathbb{N} \times \mathbb{N}) &\longrightarrow B(\ell_2) \\ [x_{ij}] &\longrightarrow [x_{ij}\varphi_{ij}] \end{aligned}$$

is obviously bounded with $\|\mathcal{M}_\varphi\| = \|[[\varphi_{ij}]]\|_{B(\ell_2)}$. Actually, more generally, if $x_{ij} \in B(H)$ with $\|x_{ij}\| \leq 1$, then the matrix $[\varphi_{ij}x_{ij}]$ defines a bounded operator on $\ell_2(H)$ with norm easily seen to be majorized by $\|[[\varphi_{ij}]]\|_{B(\ell_2)} \leq \|[[\varphi_{ij}]]\|_{B(\ell_2)}$. This shows that $\|\mathcal{M}_\varphi\|_{cb} \leq \|[[\varphi_{ij}]]\|_{B(\ell_2)}$. Let J be as above. Clearly we have

$$\|J\|_{cb} = 1$$

and hence

$$M_\varphi = \mathcal{M}_\varphi J$$

with $\|\mathcal{M}_\varphi\|_{cb} = \|[\varphi_{ij}]\|_{B(\ell_2)}$. But since $\ell_\infty(\mathbb{N} \times \mathbb{N})^{op}$ and $\ell_\infty(\mathbb{N} \times \mathbb{N})$ are identical we can factorize M_φ as follows

$$M_\varphi: B(\ell_2) \xrightarrow{J} \ell_\infty(\mathbb{N} \times \mathbb{N}) = \ell_\infty(\mathbb{N} \times \mathbb{N})^{op} \xrightarrow{\mathcal{M}_\varphi} B(\ell_2)^{op}$$

it follows that

$$\|M_\varphi: B(\ell_2) \rightarrow B(\ell_2)^{op}\|_{cb} \leq \|\mathcal{M}_\varphi\|_{cb} \leq \|[\varphi_{ij}]\|_{B(\ell_2)}.$$

This proves the “if” part.

Conversely, assume that M_φ is completely co-bounded with $\|M_\varphi\|_{cob} \leq 1$. Let $x = [x_{ij}]$ be an $n \times n$ matrix viewed as sitting in $B(\ell_2)$. Let $B = B(\ell_2)$. Note that

$$\left\| \sum_{ij=1}^n e_{ij} \otimes e_{ij} x_{ij} \right\|_{M_n(B)} = \|x\|_B$$

while

$$\left\| \sum_{ij=1}^n e_{ij} \otimes e_{ij} x_{ij} \right\|_{M_n(B^{op})} = \left\| \sum_{ij=1}^n e_{ji} \otimes e_{ij} x_{ij} \right\|_{M_n(B)} = \sup_{ij} |x_{ij}|.$$

By definition of $\|M_\varphi\|_{cob} \leq 1$, we have

$$\left\| \sum e_{ij} \otimes e_{ij} \varphi_{ij} x_{ij} \right\|_{M_n(B)} = \left\| \sum e_{ji} \otimes e_{ij} \varphi_{ij} x_{ij} \right\|_{M_n(B^{op})} \leq \left\| \sum e_{ji} \otimes e_{ij} x_{ij} \right\|_{M_n(B)}$$

which yields

$$\|[\varphi_{ij} x_{ij} 1_{\{i,j \leq n\}}]\|_B \leq \sup_{ij} |x_{ij}| \leq 1.$$

This implies

$$\|[\varphi_{ij}]_{ij \leq n}\| \leq 1$$

and since n is arbitrary we obtain

$$\|[\varphi_{ij}]\|_B \leq 1.$$

This proves the “only if” part. The proof also yields (2). \square

Corollary 5. *A Schur multiplier M_φ is completely co-bounded on $B(\ell_2)$ iff it factors through a commutative C^* -algebra or iff it factors through a minimal operator space and the corresponding factorization norm coincides with $\|M_\varphi\|_{cob}$.*

Proof. For any commutative C^* -algebra C or for any $E \subset C$, we have clearly $E = E^{op}$, so a cb -factorization $M_\varphi: B \xrightarrow{u_1} E \xrightarrow{u_2} B$ yields

$$\|M_\varphi: B \rightarrow B^{op}\|_{cb} \leq \inf\{\|u_1\|_{cb}\|u_2\|_{cb}\}$$

where the infimum runs over all possible factorizations. Conversely, the preceding shows the converse with a factorization through $\ell_\infty(\mathbb{N} \times \mathbb{N})$. \square

Remark 6. Let G be an infinite discrete group. Consider a function $f: G \rightarrow \mathbb{C}$, and the function \hat{f} defined on $G \times G$ by $\hat{f}(s, t) = f(st^{-1})$. By the well known Kesten-Hulanicki criterion (cf. e.g. [7, Th. 2.4]) G is amenable iff there is a constant C such that for any finitely supported f we have $\sum_{t \in G} |f(t)| \leq C \|\hat{f}(s, t)\|_{B(\ell_2(G))}$ and when G is amenable this holds with $C = 1$. Thus, by Theorem 4, the inequality

$$\sum_{t \in G} |f(t)| \leq C \|M_{\hat{f}}\|_{cob}$$

characterizes amenable groups. This should be compared with Bożejko's and Wysoczanski's criteria described in [7, p. 54] and [7, p. 38].

We now generalize Theorem 4 to the Schur multipliers that are bounded on the Schatten p -class S_p . We assume S_p equipped with the “natural” operator space structure introduced in [6] using the complex interpolation method. We will use freely the notation and results from [6].

Theorem 7. *Let $2 \leq p \leq \infty$. Let $T: S_p \rightarrow S_p$ denote again the transposition mapping $x \rightarrow {}^t x$. Then a bounded Schur multiplier $M_\varphi: S_p \rightarrow S_p$ is completely co-bounded iff it admits a factorization as follows:*

$$S_p \xrightarrow{J_p} \ell_p(\mathbb{N} \times \mathbb{N}) \xrightarrow{\mathcal{M}_\varphi} S_p$$

where J_p is the natural (completely contractive) inclusion and where $\|M_\varphi\|_{cob} = \|\mathcal{M}_\varphi\|_{cb}$.

Proof. Note that the fact that $J_p: S_p \rightarrow \ell_p(\mathbb{N} \times \mathbb{N})$ is completely contractive is immediate by interpolation between the cases $p = 2$ and $p = \infty$.

The proof can then be completed following the same idea as for Theorem 4. We have for any $[x_{ij}]$ in $M_n(S_p)$

$$\left\| \sum e_{ij} \otimes e_{ji} \otimes x_{ij} \right\|_{S_p^n[S_p^n[S_p]]} = \left(\sum_{ij} \|x_{ij}\|_{S_p}^p \right)^{1/p}$$

while

$$\left\| \sum e_{ij} \otimes e_{ij} \otimes x_{ij} \right\|_{S_p^n[S_p^n[S_p]]} = \|[x_{ij}]\|_{S_p^n[S_p]}.$$

Both of these identities can be proved by routine interpolation arguments starting from $p = \infty$ and $p = 2$. This gives us

$$\|[\varphi_{ij} x_{ij}]\|_{S_p^n[S_p]} \leq \|M_\varphi\|_{cob} \left(\sum \|x_{ij}\|_{S_p}^p \right)^{1/p}$$

which means (cf. [6]) that

$$\|\mathcal{M}_\varphi\|_{cb} \leq \|M_\varphi\|_{cob}.$$

To prove the converse, it suffices to notice again that

$$\ell_p(\mathbb{N} \times \mathbb{N})^{op} = \ell_p(\mathbb{N} \times \mathbb{N}). \quad \square$$

Remark 8. Consider Schur multipliers from $B(\ell_2)$ (or the subalgebra of compact operators K) into the trace class S_1 . We refer to [9] for a detailed discussion of when such a multiplier is bounded and when it is c.b. From that discussion follows easily that such a multiplier is completely co-bounded iff it is bounded. Indeed, more generally (see [9, 3]), if A, B are C^* -algebras a linear map $u: A \rightarrow B^*$ is completely co-bounded iff there are a constant c and states f_1, f_2, g_1, g_2 on A, B respectively, such that for any $(a, b) \in A \times B$

$$(3) \quad |\langle u(a), b \rangle| \leq c \left((f_1(a^*a)g_1(b^*b))^{1/2} + (f_2(aa^*)g_2(bb^*))^{1/2} \right).$$

Consider a bounded Schur multiplier $\varphi = [\varphi_{ij}]$ from $B(\ell_2)$ (or K) to S_1 , where S_1 is equipped with its natural o.s.s. as the dual of K . By this we mean that $\langle M_\varphi(a), b \rangle = \sum \varphi_{ij} a_{ij} b_{ij}$. Then (see [9]) there are two nonnegative summable sequences (λ_i) and (μ_i) such that for any i, j

$$|\varphi_{ij}| \leq \lambda_i + \mu_j.$$

Then, using the Cauchy-Schwarz inequality, we obtain (3) with the states $f_1 = g_1 = (\sum \lambda_j)^{-1} \sum \lambda_j e_{jj}$ and $f_2 = g_2 = (\sum \mu_j)^{-1} \sum \mu_j e_{jj}$. This shows that M_φ is automatically completely co-bounded. See [10] for an extension to Schur multipliers from S_p to $S_{p'}$ with $2 < p < \infty$.

Remark 9. The preceding two theorems illustrate the following simple observation: Assume that a linear map $u: E \rightarrow F$ is both c.b. and completely co-bounded, then u can be completely boundedly factorized through an operator space G for which the identity map I_G is completely co-bounded with $\|I_G\|_{cob} = 1$. Indeed, we just consider $G = F \cap F^{op}$ in the sense of [6] (this means $G = F$ equipped with the o.s.s. induced by the diagonal embedding $F \subset F \oplus F^{op}$), then u can be viewed as $u: E \rightarrow G \rightarrow F$, with $\|u: E \rightarrow G\|_{cb} = \max\{\|u\|_{cb}, \|u\|_{cob}\}$ and $\|G \rightarrow F\|_{cb} = 1$.

Conversely, any mapping of the form $u: E \xrightarrow{v} G \xrightarrow{w} F$, with c.b. maps v, w and G such that the identity $I = I_G$ on G is completely co-bounded, must be both c.b. and completely co-bounded (since $u: E \xrightarrow{v} G \xrightarrow{I} G^{op} \xrightarrow{w} F^{op}$ is c.b.).

Let us say that an operator space G is self-transposed if I_G is completely co-bounded. This property passes obviously to subspaces, quotients (and hence subquotients) and dual spaces. It is also stable under ultraproducts. Examples include any commutative C^* -algebra (or any minimal operator space), by duality any L_1 -space (or any maximal operator space) and by interpolation any L_p -space ($1 \leq p \leq \infty$). Perhaps there is a nice characterization of self-transposed operator spaces?

Let G be a finite group. Let $f, g: G \rightarrow \mathbb{C}$ be functions on G and let $\lambda(f): x \rightarrow f * x$ and $\rho(g): x \rightarrow x * g$ be the associated convolutors on $\ell_2(G)$.

The Fourier transform of f is defined as follows: for any irreducible representation π on G (i.e. $\pi \in \hat{G}$)

$$\hat{f}(\pi) = \int f(t) \pi(t)^* dm(t)$$

where m is the normalized Haar measure on G . We have then

Proposition. *With the above notation, we have*

$$\|\lambda(f)\rho(g)\|_{cob} = \sup_{\pi \in \hat{G}} \|\hat{f}(\pi)\|_2 \|\hat{g}(\pi)\|_2$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm on H_π . In particular,

$$\|Id: C^*(G) \rightarrow C^*(G)\|_{cob} = \sup_{\pi \in \hat{G}} \dim(\pi).$$

Proof. Passing to Fourier transforms, we see that $\lambda(f)\rho(g)$ coincides with $\bigoplus_{\pi \in \hat{G}} L(\hat{f}(\pi))R(\hat{g}(\pi))$ where

$L(a)$ (resp. $R(a)$) denotes left (resp. right) multiplication by a on H_π . Thus the result follows from the next lemma. \square

Lemma. *Let H be a Hilbert space and let $u: B(H) \rightarrow B(H)$ be defined by $u(x) = axb$. Then u is completely co-bounded iff a, b are both Hilbert-Schmidt operators and $\|u\|_{cob} = \|a\|_2 \|b\|_2$.*

Proof. We may easily reduce this to the finite dimensional case. So we assume $B(H) = M_n$. Then again we can write

$$\left\| \sum e_{ij} \otimes a e_{ij} b \right\| \leq \|u\|_{cob} \left\| \sum e_{ji} \otimes e_{ij} \right\| = \|u\|_{cob}.$$

But now, $T = \sum e_{ij} \otimes ae_{ij}b$ is a rank one operator on $\ell_2^n \otimes \ell_2^n = \ell_2^n(H)$ with $H = \ell_2^n$. Indeed, for any $h = (h_i)$, $k = (k_j) \in \ell_2^n(H)$ we have:

$$\begin{aligned} \langle Th, k \rangle &= \sum_{ij} \langle ae_{ij}bh_j, k_i \rangle \\ &= \sum_i \langle ae_i, k_i \rangle \sum_j \langle h_j, b^*e_j \rangle \\ &= \langle (ae_i), k \rangle \langle h, (b^*e_j) \rangle \end{aligned}$$

and hence

$$\|T\| = \left(\sum \|ae_i\|^2 \right)^{1/2} \left(\sum \|b^*e_j\|^2 \right)^{1/2} = \|a\|_2 \|b\|_2. \quad \square$$

Let us denote by $C_\lambda^*(G)$ the reduced C^* -algebra of a discrete group G , i.e. the C^* -algebra generated by the left regular representation $\lambda: G \rightarrow B(\ell_2(G))$. By a Herz-Schur multiplier on $C_\lambda^*(G)$, we mean a bounded linear map T on $C_\lambda^*(G)$ for which there is a function $f: G \rightarrow \mathbb{C}$ such that, for any $t \in G$

$$T(\lambda(t)) = f(t)\lambda(t).$$

We then denote $T_f = T$.

Corollary 10. *If G is a finite group and f is the function constantly equal to 1, then $\|T_f\|_{cob}$ (i.e. the identity map on $C_\lambda^*(G)$) is equal to the supremum of the dimensions of the irreducible representations of G .*

Remark 11. By Theorem 4, if $f \equiv 1$ as above, then the function defined on $G \times G$ by $\hat{f}(s, t) = f(st^{-1})$ (that is also constantly equal to 1), satisfies (when viewed as a Schur multiplier on $B(\ell_2(G))$) $\|M_{\hat{f}}\|_{cob} = |G|$ and in general this is different from $\|T_f\|_{cob} = \sup\{\dim(\pi) \mid \pi \in \hat{G}\}$.

Now let G be an infinite discrete group. By Remarks 2 and 3, the identity map on $A = C_\lambda^*(G)$ is completely co-bounded iff the same is true for A^{**} , and this implies that the latter is a direct sum of finitely many algebras of the form $M_n \otimes A_n$, with A_n commutative. In particular, of course A^{**} is injective, A is nuclear and hence G is amenable.

It would be interesting to describe the completely co-bounded Herz-Schur multipliers on the reduced C^* -algebra of a discrete group G in analogy with what is known for the c.b. ones (see [1]).

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